Geometrical Patterns by Mirror Reflections

Advanced Writing Skills for Graduate Study in Mathematics

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Introduction

Figure 1: *Kaleidoscope*

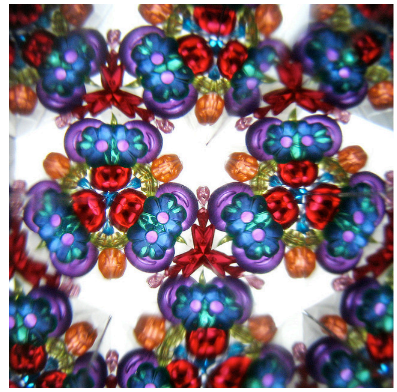


Figure 2: *Kaleidoscope Pattern*

Kaleidoscope is a tubular optical instrument with mirrors as displayed in Figure 1 [1]. Mirrors are stuck on the inside of the cylinder, and colored transparent materials like beads are put on the bottom. We look into the tube from one side and enjoy patterns of reflections. As presented in Figure 2[2], the colored materials are repeatedly reflected by the mirrors, and they show beautiful patterns.

The kaleidoscope patterns have some mathematical properties. Understanding the mathematics makes patterns more interestable. Also, we can develop different patterns using mathematical operations. For instance, we can draw another type of beautiful kaleidoscope patterns considering mirror reflection through circles.

In this paper, we will explain properties of kaleidoscope patterns and mirror reflections from a mathematical viewpoint. Moreover, we introduce mirror reflections through circles and show its kaleidoscope patterns. We use a computer to draw patterns. It will be easier to understand the mechanism of generations of the images.

Mirror Reflection

First of all, we introduce basic properties of mirror reflection. We show simplified mirror model rendered by computer. See Figure 3. There is an orange region representing a mirror and an image

[1] Photo by CameliaTWU on Visual Hunt / CC BY-NC-ND

[2] Photo by DanBrady on VisualHunt / CC BY-NC-ND

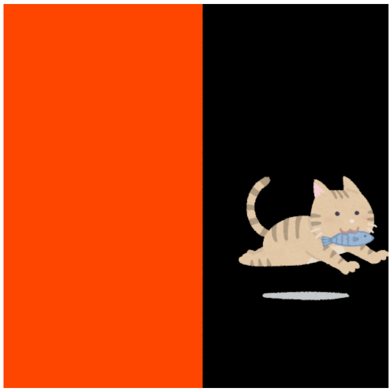


Figure 3: *Orange mirror and a cat*

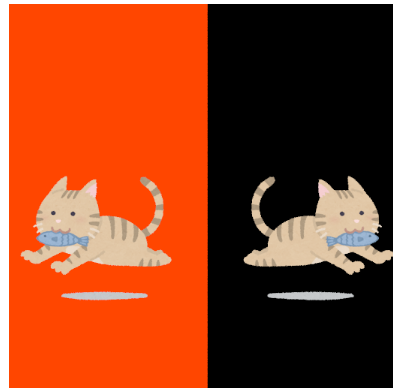


Figure 4: *Reflected cat*

of a cat. As we can see in Figure 4, the reflection through the boundary of the mirror moves a cat into the interior part of the mirror. The orientation of the cat is reversed, but the distance between the mirror and the cat is kept. The reflections preserve angles and length of the objects.

Figure 5: *Faced mirrors*

Figure 6: *Rotational symmetry*

Figure 7: *Self-intersected cat*

Next, we put one more mirror on the opposed side as presented in Figure 5. The cat is repeatedly reflected by two mirrors as indicated by yellow arrows. When we put the mirror at an angle, the reflection patterns have rotational symmetry. Figure 6 shows two mirrors crossing at $\pi/4$ and reflected images of the cat.

At this point, in order to get proper symmetrical images, we have to consider one important condition; The angle between two mirrors should be a *rational angle*, that is π/n for natural number n. If the angle is not rational, the repeated pattern will overlap each other as shown in Figure 7, because total angle around the intersection point of two mirrors should be π.

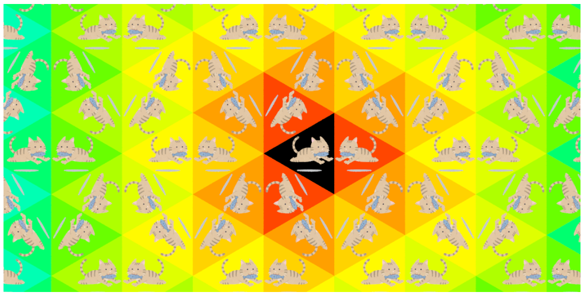


Figure 8: *All of the crossing angles is* $\pi/3$

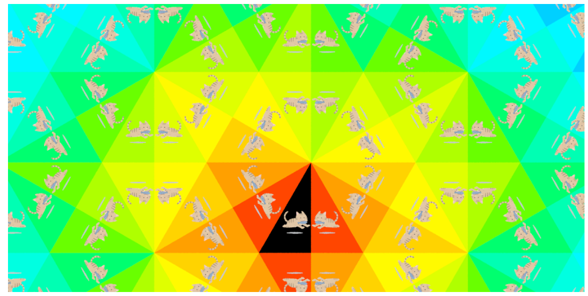


Figure 9: $\pi/2$, $\pi/3$, $\pi/6$

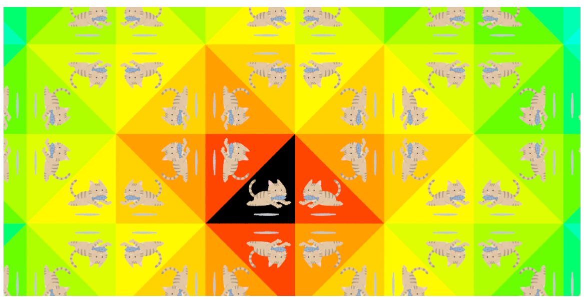


Figure 10: $\pi/2$, $\pi/4$, $\pi/4$

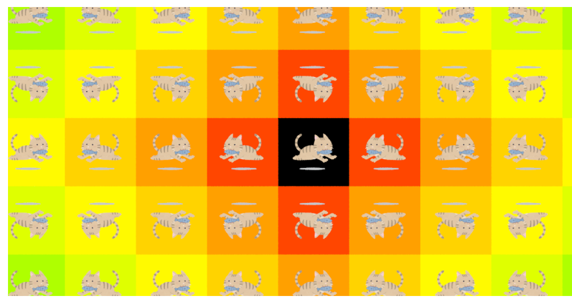


Figure 11: *All of the crossing angle is* $\pi/2$

In many cases, a kaleidoscopes has three mirrors forming a triangle. The reflections through them generate infinitely repeated kaleidoscope patterns. As described in the previous section, the angles between two mirrors should be the rational angle (π/n) to generate proper patterns. Thus, all of the interior angles of the triangle should be the rational angle.

There are only three types of triangles meeting the condition, because the sum of the interior angles of a triangle is π. The combinations of angles are as follows: ($\pi/3$, $\pi/3$, $\pi/3$), ($\pi/2$, $\pi/3$, $\pi/6$), and ($\pi/2$, $\pi/4$, $\pi/4$). The images of kaleidoscope patterns are shown in Figure 8, Figure 9, and Figure 10. Also, there are rectangular kaleidoscope patterns generated by four mirrors. See Figure 11. All of the angles should be $\pi/2$.

We can see triangles or rectangles are closely tiled in the figures. Of course, we can make kaleidoscopes whose mirrors don't satisfy the conditions of angles. Their patterns maybe still beautiful although the symmetry of the figure is broken. However, mathematicians pursue the patterns which spread all over with no gaps and self-intersections. For more details about the properties of the images, read chapter 2 of *The Symmetries of Things* [1].

We dealt with mirror reflections through the line so far. In the following sections, we introduce mirror reflections through the curved surface, that is circles. Reflections through circles bring complexity to the patterns. We can not make such kaleidoscope patterns in reality, but they can be visualized by computer.

Mirror Reflection through the Circle

Figure 12: *Apply Circle Inversion to an image of a cat*

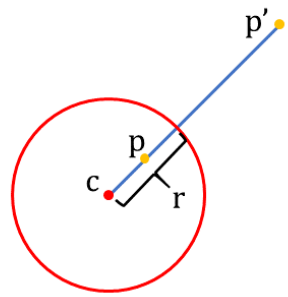


Figure 13: *p' is the inverse of p*

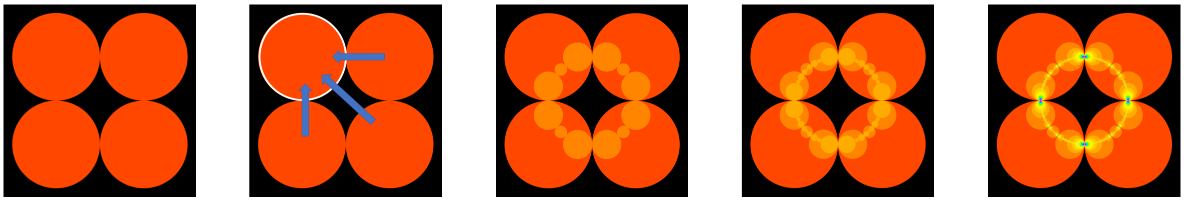
Firstly, we introduce a mirror reflection through a circle. It is also known as *circle inversion*. See Figure 12. There are an orange disk and an image of a cat. The reflection through the boundary of the disk move the cat into the interior of the circle. The reflected cat is deformed, but it preserves angle of the shapes. For instance, the cat's tail has similar curves before and after reflections.

Mathematically, the reflection through the circle with center c and radius r moves the point p into p' such that

$$|cp| \cdot |cp'| = r^2.$$

It is presented in Figure 13. The reflection swaps infinity point and center of the circle. For more details about the mirror reflection through circles, see the chapter 2 of the book, *Indra's Pearls* [2].

Note that, a circle with infinitely large radius is considered as a line. Thus, we can say that all of the kaleidoscope patterns are generated by reflections through circles.



(a) (b) (c) (d) (e)

Figure 14: *The process of the mirror reflections through circles*

(a) (b)

Figure 15: *Cat reflected by four circles*

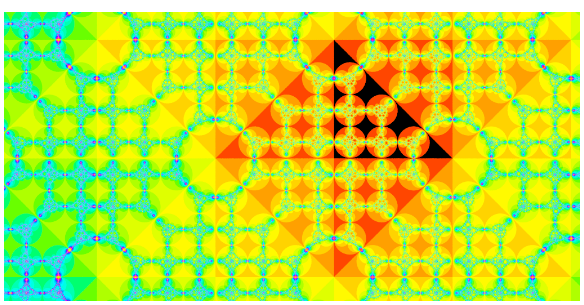


Figure 16: *Space-filling kaleidoscope pattern 1*

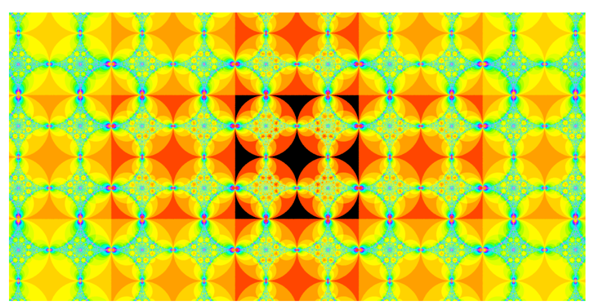


Figure 17: *Space-filling kaleidoscope pattern 2*

We show some examples of kaleidoscope patterns generated by reflections through circles. In Figure 14, we show the process of the generation of the kaleidoscope composed of four circular mirror reflections. As presented in Figure 14(a), we assume there are four orange disks and we consider reflections through their boundaries. Now we focus on the white circle in Figure 14(b).

The reflection through the circle moves the outer three disks into the interior of the white circle. So, all of the reflections in the circles generate twelve small disks as presented in Figure 14(c). The smaller disks are also reflected by the four original circles. Then there are more smaller reflected circles as shown in Figure 14(d). The reflections are infinitely repeated, and finally, we get Figure 14(e). There are infinite number of reflected circles.

In the similar manner to usual kaleidoscope pattern, we can draw the pattern of a cat. In Figure 15a, we can see actions of the mirror reflections. The cats are repeatedly reflected by circles. We can draw more complicated space-filling patterns composed of planar mirrors and circular mirrors in Figure 16 and Figure 17.

This kind of patterns with circles are studied by Robert Fricke and Felix Klein [3] in the 19th century. They and their students drew the images by hand. However, it was difficult to visualize complicated arrangements. After computer became popular, Benoit Mandelbrot, a French mathematician who studied infinitely repeated shapes called fractals studied and visualized them by computer [4].

Summary

In this paper, we showed mathematical properties of kaleidoscope patterns. There is a requirement about angles between two mirrors to generate a proper pattern. Therefore only three types of triangles can form a good kaleidoscope. We also considered mirror reflection through circles, and we made more complicated and beautiful reflected patterns.

References

[1] J.H. Conway, H. Burgiel, and C. Goodman-Strauss. *The Symmetries of Things*. Ak Peters Series. Taylor & Francis, 2008. ISBN: 9781568812205.

[2] David Mumford, Caroline Series, and David Wright. *Indra's Pearls: The Vision of Felix Klein*. 1st ed. Cambridge University Press, Apr. 2002.

[3] R. Fricke and F. Klein. *Vorlesungen über die Theorie der automorphen Funktionen: Die gruppentheoretischen Grundlagen*. Bibliotheca mathematica Teubneriana 1. B.G. Teubner, 1965. URL: `https://books.google.co.jp/books?id=P2VbAAAAcAAJ`.

[4] Benoit B Mandelbrot. *The fractal geometry of nature*. WH freeman New York, 1983.